



**Theorem 8.2.1** If  $T : V \rightarrow W$  is a linear transformation, then  $T$  is one-to-one if and only if  $\ker(T) = \{0\}$ .

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**#4** Determine whether the linear transformation is one-to-one by finding its kernel and then applying Theorem 8.2.1.

- a.  $T : R^2 \rightarrow R^3$ , where  $T(x, y) = (x - y, y - x, 2x - 2y)$   
b.  $T : R^2 \rightarrow R^2$ , where  $T(x, y) = (0, 2x + 3y)$   
c.  $T : R^2 \rightarrow R^2$ , where  $T(x, y) = (x + y, x - y)$
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**#5** Determine whether multiplication by  $A$  is one-to-one by computing the nullity of  $A$  and then applying Theorem 8.2.1

a.  $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$

b.  $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 7 & 2 & 4 \\ -1 & -3 & 0 & 0 \end{bmatrix}$

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**Theorem 8.2.2** If  $V$  and  $W$  are finite-dimensional vector spaces with the same dimension, and if  $T : V \rightarrow W$  is a linear transformation, then the following statements are equivalent.

- a)  $T$  is one-to-one.
- b)  $\ker(T) = \{0\}$ .
- c)  $T$  is onto [i.e.,  $R(T) = W$ ].

**Theorem 8.2.3** If  $T_A : R^n \rightarrow R^m$  is a matrix transformation, then

- a)  $T_A$  is one-to-one if and only if the columns of  $A$  are linearly independent.
- b)  $T_A$  is onto if and only if the columns of  $A$  span  $R^m$ .

**#14** Use Theorem 8.2.3 to determine whether multiplication by  $A$  is one-to-one, onto, both, or neither. Justify your answer.

a.  $A = \begin{bmatrix} 9 & -3 \\ -4 & 2 \\ 1 & 1 \end{bmatrix}$

b.  $A = \begin{bmatrix} 3 & -3 & 1 & 1 \\ 6 & -6 & 0 & 2 \\ 9 & -9 & 1 & 3 \end{bmatrix}$



- j) The column vectors of  $A$  span  $R^n$ .
- k) The row vectors of  $A$  span  $R^n$ .
- l) The column vectors of  $A$  form a basis for  $R^n$ .
- m) The row vectors of  $A$  form a basis for  $R^n$ .
- n)  $A$  has rank  $n$ .
- o)  $A$  has nullity 0.
- p) The orthogonal complement of the null space of  $A$  is  $R^n$ .
- q) The orthogonal complement of the row space of  $A$  is  $\{0\}$ .
- r)  $\lambda = 0$  is not an eigenvalue of  $A$ . [This is from Theorem 5.1.5.]
- s)  $A^T A$  is invertible. [This is from Theorem 6.4.5.]
- t) The kernel of  $T_A$  is  $\{0\}$ .
- u) The range of  $T_A$  is  $R^n$ .
- v)  $T_A$  is one-to-one.

**Definition:** If  $T : V \rightarrow W$  is a one-to-one linear transformation with range  $R(T)$ , and if  $\mathbf{w}$  is any vector in  $R(T)$ , then there is exactly one vector  $\mathbf{v}$  in  $V$  for which  $T(\mathbf{v}) = \mathbf{w}$ . The **inverse of T** maps  $\mathbf{w}$  back into  $\mathbf{v}$  and is denoted by  $T^{-1}$ . (This is analogous to **inverse** in Section 1.9.)

**Definition:** (analogous to **composition** in Section 1.9)

If  $T_1 : U \rightarrow V$  and  $T_2 : V \rightarrow W$  are linear transformations, then the **composition** of  $T_2$  with  $T_1$ , denoted by  $T_2 \circ T_1$  (which is read “ $T_2$  circle  $T_1$ ”) is the function defined by the formula  $(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$  where  $\mathbf{u}$  is a vector in  $U$ .

**Theorem 8.2.5** If  $T_1 : U \rightarrow V$  and  $T_2 : V \rightarrow W$  are linear transformations, then  $(T_2 \circ T_1) : U \rightarrow W$  is also a linear transformation.

**Theorem 8.2.6** If  $T_1 : U \rightarrow V$  and  $T_2 : V \rightarrow W$  are one-to-one linear transformations, then:

- a)  $T_2 \circ T_1$  is one-to-one.
- b)  $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$ .

**#38** Let  $D(\mathbf{f}) = f'(x)$  and  $J(\mathbf{f}) = \int_0^x f(t)dt$  be the derivative and integral linear transformations. Find  $(J \circ D)(\mathbf{f})$  for

a.  $f(x) = x^2 + 3x + 2$

b.  $f(x) = \sin x$

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### Key concepts from 8.3:

Definition 1: A linear transformation  $T : V \rightarrow W$  that is both one-to-one and onto is said to be an **isomorphism**, and  $W$  is said to be **isomorphic** to  $V$ .

**Theorem 8.3.1** Every real  $n$ -dimensional vector space is isomorphic to  $R^n$ . [In this sense, every  $n$ -dimensional vector space is algebraically equivalent to  $R^n$ .]

**Theorem 8.3.2** If  $S$  is an ordered basis for a vector space  $V$ , then the coordinate map  $\mathbf{u} \xrightarrow{T} (\mathbf{u})_S$  is an isomorphism between  $V$  and  $R^n$ .

**Theorem 8.3.X** A linear transformation  $T : V \rightarrow W$  is an isomorphism if and only if whenever  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  is a basis for  $V$ ,  $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$  is a basis for  $W$ .

### Examples of Natural isomorphisms

Between  $P_{n-1}$  and  $R^n$ :  $a_0 + a_1x + \dots + a_{n-1}x^{n-1} \xrightarrow{T} (a_0, a_1, \dots, a_{n-1})$

Between  $M_{22}$  and  $R^4$ :  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} (a, b, c, d)$